<u>MSE 113</u> <u>Summary of Essentials</u> (Continuum Mechanics) (Elasticity/Plasticity) R. O. Ritchie

Equilibrium

$$\frac{\partial\sigma_{11}}{\partial x_1}+\frac{\partial\sigma_{21}}{\partial x_2}+\frac{\partial\sigma_{31}}{\partial x_3}=0$$

$$\frac{\partial\sigma_{12}}{\partial x_1}+\frac{\partial\sigma_{22}}{\partial x_2}+\frac{\partial\sigma_{32}}{\partial x_3}=0 \qquad \text{assuming no body forces or acceleration}$$

$$\frac{\partial\sigma_{13}}{\partial x_1}+\frac{\partial\sigma_{23}}{\partial x_2}+\frac{\partial\sigma_{33}}{\partial x_3}=0$$

Compatibility

$$\varepsilon_{ij}=\tfrac{1}{2}\left(\frac{\partial u_i}{\partial x_j}+\frac{\partial u_j}{\partial x_i}\right)$$

Strains in Cylindrical Coordinates:

$$\varepsilon_{rr}=\frac{\partial u_r}{\partial r};\ \varepsilon_{\theta\theta}=\tfrac{1}{r}\frac{\partial u_\theta}{\partial\theta}+\frac{u_r}{r};\ \varepsilon_{zz}=\frac{\partial u_z}{\partial z}$$

$$\varepsilon_{zr}=\tfrac{1}{2}\left(\frac{\partial u_r}{\partial z}+\frac{\partial u_z}{\partial r}\right);\ \varepsilon_{r\theta}=\tfrac{1}{2}\left(\frac{\partial u_\theta}{\partial r}+\tfrac{1}{r}\frac{\partial u_r}{\partial\theta}-\frac{u_\theta}{r}\right);\ \varepsilon_{z\theta}=\tfrac{1}{2}\left(\frac{\partial u_z}{r\partial\theta}+\frac{\partial u_\theta}{\partial z}\right)$$

Strains in Spherical Coordinates:

$$\varepsilon_{rr}=\frac{\partial u_r}{\partial r};\ \varepsilon_{\phi\phi}=\tfrac{1}{r}\frac{\partial u_\phi}{\partial\phi}+\frac{u_r}{r};\ \varepsilon_{\theta\theta}=\frac{1}{r\sin\phi}\frac{\partial u_\theta}{\partial\theta}+\frac{u_r}{r}+\frac{u_\phi}{r}\cot\phi$$

$$\varepsilon_{r\theta}=\tfrac{1}{2}\left(\frac{1}{r\sin\phi}\frac{\partial u_r}{\partial\theta}+\frac{\partial u_\theta}{\partial r}-\frac{u_\theta}{r}\right);\ \varepsilon_{r\phi}=\tfrac{1}{2}\left(\frac{\partial u_\phi}{\partial r}-\frac{u_\phi}{r}+\frac{1}{r}\frac{\partial u_r}{\partial\phi}\right)$$

$$\varepsilon_{\phi\theta}=\tfrac{1}{2}\left(\tfrac{1}{r}\frac{\partial u_\theta}{\partial\phi}-\frac{u_\theta}{r}\cos\phi+\frac{1}{r\sin\phi}\frac{\partial u_\phi}{\partial\theta}\right)$$

Definitions

$$\text{Hydrostatic stress: } \sigma = \tfrac{1}{3}\sum_{i=1}^{3}\sigma_{ii} \qquad \text{Deviatoric stress: } \sigma'_{ij} = \sigma_{ij} - \delta_{ij}\,\sigma \quad \delta_{ij} = 1 \text{ for } i = j$$

$$\text{Dilation: } \varepsilon = \sum_{i=1}^{3}\varepsilon_{ii} \approx \Delta V / V \qquad \text{Deviator strain: } \varepsilon'_{ij} = \varepsilon_{ij} - \delta_{ij}\,\varepsilon/3 \quad \delta_{ij} = 0 \text{ for } i \neq j$$

$$\bar{\sigma} = \left\{\tfrac{1}{2}\left[(\sigma_{11}-\sigma_{22})^2 + (\sigma_{22}-\sigma_{33})^2 + (\sigma_{33}-\sigma_{11})^2\right] + 3\left(\sigma_{12}^2 + \sigma_{23}^2 + \sigma_{31}^2\right)\right\}^{1/2}$$

$$\left(d\bar{\varepsilon}_p\right)^2 = \tfrac{4}{9}\left\{\tfrac{1}{2}\left[\left(d\varepsilon_{11}^p - d\varepsilon_{22}^p\right)^2 + \left(d\varepsilon_{22}^p - d\varepsilon_{33}^p\right)^2 + \left(d\varepsilon_{33}^p - d\varepsilon_{11}^p\right)^2\right] + 3\left(d\varepsilon_{12}^{p2} + d\varepsilon_{23}^{p2} + d\varepsilon_{31}^{p2}\right)\right\}$$

Constitutive Relations

Elasticity

$$\varepsilon_{11} = \tfrac{1}{E}\left[\sigma_{11} - \nu(\sigma_{22}+\sigma_{33})\right]; \quad \varepsilon_{12} = \sigma_{12}/2G$$

$$\varepsilon_{22} = \tfrac{1}{E}\left[\sigma_{22} - \nu(\sigma_{33}+\sigma_{11})\right]; \quad \varepsilon_{23} = \sigma_{23}/2G$$

$$\varepsilon_{33} = \tfrac{1}{E}\left[\sigma_{33} - \nu(\sigma_{22}+\sigma_{11})\right]; \quad \varepsilon_{31} = \sigma_{31}/2G$$

$$G = E/2(1+\nu) \qquad B = E/3(1-2\nu) = \sigma/\varepsilon$$

Plasticity

$$d\varepsilon_{11}^p = \frac{d\bar{\varepsilon}_p}{\bar{\sigma}}\left[\sigma_{11} - \tfrac{1}{2}(\sigma_{22}+\sigma_{33})\right]; \quad d\varepsilon_{12}^p = \frac{3}{2}\frac{d\bar{\varepsilon}^p}{\bar{\sigma}}\sigma_{12}$$

$$d\varepsilon_{22}^p = \frac{d\bar{\varepsilon}_p}{\bar{\sigma}}\left[\sigma_{22} - \tfrac{1}{2}(\sigma_{11}+\sigma_{33})\right]; \quad d\varepsilon_{23}^p = \frac{3}{2}\frac{d\bar{\varepsilon}^p}{\bar{\sigma}}\sigma_{23}$$

$$d\varepsilon_{33}^p = \frac{d\bar{\varepsilon}_p}{\bar{\sigma}}\left[\sigma_{33} - \tfrac{1}{2}(\sigma_{11}+\sigma_{22})\right]; \quad d\varepsilon_{31}^p = \frac{3}{2}\frac{d\bar{\varepsilon}^p}{\bar{\sigma}}\sigma_{31}$$

$$\text{i.e., } d\varepsilon_{ij}'^{p} = \tfrac{3}{2}\frac{d\bar{\varepsilon}^p}{\bar{\sigma}}\sigma'_{ij}$$

Yield Criteria

$$\tau_{max} = k; \quad \bar{\sigma} = Y$$